Generalised Criteria on Delay Dependent Stability of Highly Nonlinear Hybrid Stochastic Systems

Weiyin Fei[1], **Liangjian Hu**[2,*], **Xuerong Mao**[3], **Mingxuan Shen**[1,4]
[1] School of Mathematics and Physics,
Anhui Polytechnic University, Wuhu, Anhui 241000, China.
[2] Department of Applied Mathematics,
Donghua Univerisity, Shanghai 201620, China.
[3] Department of Mathematics and Statistics,
University of Strathclyde, Glasgow G1 1XH, U.K.
[4] School of Science, Nanjing University of Science and Technology,
Nanjing, Jiangsu 210094, China.

Abstract

Our recent paper [2] is the first to establish delay dependent criteria for highly nonlinear hybrid stochastic differential delay equations (SDDEs) (by highly nonlinear we mean the coefficients of the SDDEs do not have to satisfy the linear growth condition). This is an important breakthrough in the stability study as all existing delay stability criteria before could only be applied to delay equations where their coefficients are either linear or nonlinear but bounded by linear functions (namely, satisfy the linear growth condition). In this continuation, we will point out one restrictive condition imposed in our earlier paper [2]. We will then develop our ideas and methods there in order to remove this restrictive condition so that our improved results cover a much wider class of hybrid SDDEs.

Key words: Hybrid delay systems, Itô's formula, almost sure asymptotic stability, Lyapunov functional.

1 Introduction

There are huge numbers of papers on the stability of delay systems. For example, [8, 12] are two of the best books on the stability of differential delay equations (DDEs) while [31] contains a nice literature review. In the area of stability of stochastic differential delay equations (SDDEs), we mention five books [5, 13, 14, 15, 23] among others. On the other hand, many real-world systems are often disturbed by abrupt events so their structures will be changed abruptly. Hybrid systems have therefore developed intensively (see, e.g., [4, 6, 7, 28, 32, 33, 34]). Among these references, the sliding mode control of discrete-time switched systems and multi-area power systems is analysed in [32, 33], respectively; the fault detection filter design for non-homogeneous Markovian jump systems is investigated by a Takagi-Sugeno fuzzy approach in [6]; while [28] considers the exponential passive filtering for a class of stochastic neutral-type neural networks with both semi-Markovian jump parameters and mixed time delays. One of the important classes of hybrid systems is the class of hybrid SDDEs (also known as SDDEs with Markovian switching), which have been developed very quickly for the past twenty years to model real-world systems where they may experience abrupt changes in their structure and parameters in addition to time delays and uncertainties. One of the important issues in the study of hybrid SDDEs is the stability analysis (see, e.g., [3, 10, 11, 16, 20, 22, 26, 27, 29, 30, 35, 36, 40, 41, 42]),

*Corresponding author. E-mail: ljhu@dhu.edu.cn

and the robust stability of nonlinear hybrid delay systems with different frameworks is discussed in, e.g., [3, 9, 27]

The stability criteria are often classified into two categories: delay-dependent and delay-independent stability criteria (see, e.g., [1, 17, 18, 24, 25]). The delay-independent stability criteria work for any size of delays, while the delay-dependent stability criteria take into account the size of delays and hence are generally less conservative than the delay-independent ones (see, e.g., [19, 37, 38, 39]). A common feature of these existing delay-dependent stability criteria is that they can only be applied to hybrid SDDEs where their coefficients are either linear or nonlinear but bounded by linear functions (namely, satisfy the linear growth condition). Our recent paper [2] is the first to establish delay dependent criteria for highly nonlinear hybrid SDDEs (by highly nonlinear we mean the coefficients of the SDDEs do not have to satisfy the linear growth condition). This is an important breakthrough in the stability study.

However, there is a restrictive condition imposed in [2]. To point out this, let us recall the underlying hybrid SDDE

$$dx(t) = f(x(t), x(t-\delta(t)), r(t), t)dt + g(x(t), x(t-\delta(t)), r(t), t)dB(t) \tag{1.1}$$

considered there. The notation used will be explained in Section 2. Assumption 3.3 in [2] requires the drift coefficient $f : R^n \times R^n \times S \times R_+ \to R^n$ to be globally Lipschtiz continuous in the second variable (the delay component), namely there is a positive constant β_4 such that

$$|f(x, x, i, t) - f(x, y, i, t)| \le \beta_4 |x - y|, \quad (x, y, i, t) \in R^n \times R^n \times S \times R_+.$$

This excludes many highly nonlinear hybrid SDDEs, for example, the one to be discussed in Example 4.1 where

$$f(x, y, 1, t) = -y + y^3 - 5x^3 \quad \text{and} \quad f(x, y, 2, t) = y - \frac{y^3}{2} - 3x^3.$$

Our aim in this paper is to remove this restrictive condition so that our generalised results cover a much wider class of hybrid SDDEs. Let us begin to develop our generalised results.

2 Preliminary

We will use the same notation as in [2]. However, for the convenience of the reader, we repeat here. If A is a vector or matrix, its transpose is denoted by A^T. If $x \in R^n$, then $|x|$ is its Euclidean norm. If A is a matrix, we let $|A| = \sqrt{\text{trace}(A^T A)}$ be its trace norm and $\|A\| = \max\{|Ax| : |x| = 1\}$ be the operator norm. If A is a symmetric matrix ($A = A^T$), denote by $l_{\min}(A)$ and $l_{\max}(A)$ its smallest and largest eigenvalue, respectively. Let $R_+ = [0, \infty)$. For $h > 0$, denote by $C([-h, 0]; R^n)$ the family of continuous functions φ from $[-h, 0] \to R^n$ with the norm $\|\varphi\| = \sup_{-h \le u \le 0} |\varphi(u)|$. If both a, b are real numbers, then $a \wedge b = \min\{a, b\}$ and $a \vee b = \max\{a, b\}$.

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\ge0}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\ge0}$ satisfying the usual conditions (i.e. it is increasing and right continuous while $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets). If A is a subset of Ω, denote by I_A its indicator function; that is $I_A(\omega) = 1$ if $\omega \in A$ and 0 otherwise. Let $B(t) = (B_1(t), \cdots, B_m(t))^T$ be an m-dimensional Brownian motion defined on the probability space. Let $r(t)$, $t \ge 0$, be a right-continuous Markov chain on the probability space taking values in a finite state space $S = \{1, 2, \cdots, N\}$ with generator $\Gamma = (\gamma_{ij})_{N\times N}$ given by

$$\mathbb{P}\{r(t+\Delta) = j | r(t) = i\} = \begin{cases} \gamma_{ij}\Delta + o(\Delta) & \text{if } i \ne j, \\ 1 + \gamma_{ii}\Delta + o(\Delta) & \text{if } i = j, \end{cases}$$

where $\Delta > 0$. Here $\gamma_{ij} \geq 0$ is the transition rate from i to j if $i \neq j$ while

$$\gamma_{ii} = -\sum_{j \neq i} \gamma_{ij}.$$

We assume that the Markov chain $r(\cdot)$ is independent of the Brownian motion $B(\cdot)$.

Let $C^{2,1}(R^n \times S \times R_+; R_+)$ denote the family of non-negative functions $U(x, i, t)$ defined on $(x, i, t) \in R^n \times S \times R_+$ which are continuously twice differentiable in x and once in t. For such a function $U(x, i, t)$, we will let

$$U_t(x, i, t) = \frac{\partial U(x, i, t)}{\partial t}, \quad U_x(x, i, t) = \Big(\frac{\partial U(x, i, t)}{\partial x_1}, \cdots, \frac{\partial U(x, i, t)}{\partial x_n}\Big),$$

and

$$U_{xx}(x, i, t) = \Big(\frac{\partial^2 U(x, i, t)}{\partial x_k \partial x_l}\Big)_{n \times n}.$$

Let $C(R^n \times [-\tau, \infty); R_+)$ denote the family of all continuous functions from $R^n \times [-\tau, \infty)$ to R_+.

Let $\tau > 0$ and $\bar{\delta} \in [0, 1)$ be two constants. Let δ be a differentiable function from $R_+ \to [0, \tau]$ such that $\dot{\delta}(t) := d\delta(t)/dt \leq \bar{\delta}$ for all $t \geq 0$. Let

$$f : R^n \times R^n \times S \times R_+ \to R^n \quad \text{and} \quad g : R^n \times R^n \times S \times R_+ \to R^{n \times m}$$

be Borel measurable functions. Consider an n-dimensional hybrid SDDE

$$dx(t) = f(x(t), x(t - \delta(t)), r(t), t)dt + g(x(t), x(t - \delta(t)), r(t), t)dB(t) \tag{2.1}$$

on $t \geq 0$ with initial data

$$\{x(t) : -\tau \leq t \leq 0\} = \xi \in C([-\tau, 0]; R^n) \text{ and } r(0) = i_0 \in S. \tag{2.2}$$

As a *standing hypothesis* of this paper, we assume that both coefficients f and g are sufficiently smooth so that the SDDE (2.1) with the initial data (2.2) has the unique global solution $x(t)$ on $t \geq -\tau$ and, moreover, there is a constant $q \geq 2$ such that

$$\sup_{-\tau \leq t < \infty} \mathbb{E}|x(t)|^q < \infty. \tag{2.3}$$

For example, Assumptions 2.1 and 2.2 in [2] are sufficient for these while we refer the reader to [9] for further information.

The Lyapunov functional defined in [2] will play its key role in this paper again. As in [2], we define two segments $\hat{x}_t := \{x(t + s) : -2\tau \leq s \leq 0\}$ and $\hat{r}_t := \{r(t + s) : -2\tau \leq s \leq 0\}$ for $t \geq 0$. For $\hat{x}_t$ and $\hat{r}_t$ to be well defined for $0 \leq t < 2\tau$, we set $x(s) = \xi(-\tau)$ for $s \in [-2\tau, -\tau)$ and $r(s) = r_0$ for $s \in [-2\tau, 0)$. The Lyapunov functional has the form

$$\begin{aligned} V(\hat{x}_t, \hat{r}_t, t) &= U(x(t), r(t), t) \\ &+ \theta \int_{-\tau}^{0} \int_{t+s}^{t} \Big[\tau |f(x(v), x(v - \delta(v)), r(v), v)|^2 + |g(x(v), x(v - \delta(v)), r(v), v)|^2\Big] dv ds \end{aligned} \tag{2.4}$$

for $t \geq 0$, where $U \in C^{2,1}(R^n \times S \times R_+; R_+)$ and θ is a positive number to be determined later while we set

$$f(x, y, i, s) = f(x, y, i, 0), \quad g(x, y, i, s) = g(x, y, i, 0)$$

for $(x, y, i, s) \in R^n \times R^n \times S \times [-2\tau, 0)$. It was shown in [2] that

$$dV(\hat{x}_t, \hat{r}_t, t) = LV(\hat{x}_t, \hat{r}_t, t)dt + dM(t), \tag{2.5}$$

where $M(t)$ is a continuous local martingale with $M(0) = 0$ (the explicit form of $M(t)$ is of no use in this paper so we do not state it here but it can be found in [21, Theorem 1.45 on page 48]) and

$$\begin{aligned}
&LV(\hat{x}_t, \hat{r}_t, t) \\
&= U_t(x(t), r(t), t) + U_x(x(t), r(t), t) f(x(t), x(t-\delta(t)), r(t), t) \\
&+ \tfrac{1}{2}\text{trace}[g^T(x(t), x(t-\delta(t)), r(t), t) U_{xx}(x(t), r(t), t) g(x(t), x(t-\delta(t)), r(t), t)] \\
&+ \sum_{j=1}^{N} \gamma_{r(t),j} U(x(t), j, t) \\
&+ \theta\tau\Big[\tau|f(x(t), x(t-\delta(t)), r(t), t)|^2 + |g(x(t), x(t-\delta(t)), r(t), t)|^2\Big] \\
&- \theta\int_{t-\tau}^{t} \Big[\tau|f(x(v), x(v-\delta(v)), r(v), v)|^2 + |g(x(v), x(v-\delta(v)), r(v), v)|^2\Big] dv.
\end{aligned} \tag{2.6}$$

3 Main Results

To study the delay-dependent asymptotic stability of the SDDE (2.1), we need to impose a couple of assumptions. The first one is the polynomial growth condition.

Assumption 3.1 *Assume that there exist three constants $K > 0$, $q_1 \geq 1$ and $q_2 \geq 1$ such that*

$$|f(x, y, i, t)| \leq K(1 + |x|^{q_1} + |y|^{q_1}) \text{ and } |g(x, y, i, t)| \leq K(1 + |x|^{q_2} + |y|^{q_2}) \tag{3.1}$$

for all $(x, y, i, t) \in R^n \times R^n \times S \times R_+$. Assume also that $2(q_1 \vee q_2) \leq q$, where q is specified in our standing hypothesis (2.3).

This assumption, along with (2.3), shows that both $f(x(t), x(t-\delta(t), r(t), t)$ and $g(x(t), x(t-\delta(t), r(t), t)$ are in L^2 for all $t \geq 0$. We are now introduce a key assumption which removes the restrictive condition imposed in [2].

Assumption 3.2 *Assume that the drift coefficient f can be decomposed as*

$$f(x, y, i, t) = f_1(x, y, i, t) + f_2(x, y, i, t) \tag{3.2}$$

and, moreover, there is a positive number β_4 such that

$$|f_1(x, x, i, t) - f_1(x, y, i, t)| \leq \beta_4|x - y| \tag{3.3}$$

for all $(x, y, i, t) \in R^n \times R^n \times S \times R_+$.

Decomposition (3.2) enables us to arrange the highly nonlinear term(s) of y (the delay component) into f_2 but leave the globally-Lipschitz-continuous term(s) in f_1. Example 4.1 will illustrate this. With this assumption, we can arrange the second term on the right hand side of (2.6) as

$$\begin{aligned}
&U_x(x, i, t) f(x, y, i, t) \\
=&U_x(x, i, t)\Big([f_1(x, x, i, t) + f_2(x, y, i, t)] + [f_1(x, y, i, t) - f_1(x, x, i, t)]\Big).
\end{aligned}$$

Consequently, we can rearrange (2.6) in order to have

$$\begin{aligned}
&LV(\hat{x}_t, \hat{r}_t, t) = \mathcal{L}U(x(t), x(t-\delta(t)), r(t), t) \\
&+ U_x(x(t), r(t), t)[f_1(x(t), x(t-\delta(t)), r(t), t) - f_1(x(t), x(t), r(t), t)] \\
&+ \theta\tau\Big[\tau|f(x(t), x(t-\delta(t)), r(t), t)|^2 + |g(x(t), x(t-\delta(t)), r(t), t)|^2\Big] \\
&- \theta\int_{t-\tau}^{t} \Big[\tau|f(x(v), x(v-\delta(v)), r(v), v)|^2 + |g(x(v), x(v-\delta(v)), r(v), v)|^2\Big] dv,
\end{aligned} \tag{3.4}$$

where $\mathcal{L}U : R^n \times R^n \times S \times R_+ \to R$ is defined by

$$\mathcal{L}U(x,y,i,t) = U_t(x,i,t) + U_x(x,i,t)[f_1(x,x,i,t) + f_2(x,y,i,t)] \\ + \tfrac{1}{2}\mathrm{trace}[g^T(x,y,i,t)U_{xx}(x,i,t)g(x,y,i,t)] + \sum_{j=1}^{N} \gamma_{ij} U(x,j,t). \tag{3.5}$$

The reason why we define $\mathcal{L}U$ as above is because it is associated with the following hybrid SDDE

$$dX(t) = [f_1(X(t),X(t),r(t),t) + f_2(X(t),X(t-\delta(t)),r(t),t)]dt \\ + g(X(t),X(t-\delta(t)),r(t),t)dB(t) \tag{3.6}$$

in the sense that

$$dU(X(t),r(t),t) = \mathcal{L}U(X(t),X(t-\delta(t)),r(t),t)dt \\ + U_x(X(t),r(t),t)g(X(t),X(t-\delta(t)),r(t),t)dB(t).$$

We can now state a technical assumption.

Assumption 3.3 *Assume that there are functions $U \in C^{2,1}(R^n \times S \times R_+; R_+)$, $U_1, U_2 \in C(R^n \times [-\tau, \infty); R_+)$, and positive numbers α_k $(k = 1,2,3,4)$ and β_j $(j = 1,2,3)$ such that*

$$\alpha_2 < \alpha_1(1-\bar{\delta}), \quad \alpha_4 \le \alpha_3(1-\bar{\delta}), \tag{3.7}$$

and

$$\mathcal{L}U(x,y,i,t) + \beta_1|U_x(x,i,t)|^2 + \beta_2|f(x,y,i,t)|^2 + \beta_3|g(x,y,i,t)|^2 \\ \le -\alpha_1 U_1(x,t) + \alpha_2 U_1(y,t-\delta(t)) - \alpha_3 U_2(x,t) + \alpha_4 U_2(y,t-\delta(t)), \tag{3.8}$$

for all $(x,y,i,t) \in R^n \times R^n \times S \times R_+$.

Let us comment on this assumption. Condition (3.8) implies

$$\mathcal{L}U(x,y,i,t) \le -\alpha_1 U_1(x,t) + \alpha_2 U_1(y,t-\delta(t)) - \alpha_3 U_2(x,t) + \alpha_4 U_2(y,t-\delta(t)), \tag{3.9}$$

which guarantees the asymptotic stability (with some additional condition) of the hybrid SDDE (3.6). Re-arranging our underlying hybrid SDDE (2.1) as

$$\begin{aligned} dx(t) &= [f_1(x(t),x(t),r(t),t) + f_2(x(t),x(t-\delta(t)),r(t),t)]dt \\ &+ g(x(t),x(t-\delta(t)),r(t),t)dB(t) \\ &+ \big[f_1(x(t),x(t-\delta(t)),r(t),t) - f_1(x(t),x(t),r(t),t)\big]dt, \end{aligned} \tag{3.10}$$

we see that the SDDE (2.1) may be regarded as a perturbed system of the stable SDDE (3.6). We observe that $x(t)$ should be close to $X(t)$ if the difference

$$f_1(x(t),x(t-\delta(t)),r(t),t) - f_1(x(t),x(t),r(t),t)$$

would be sufficiently small provided the time delay is not too large. To guarantee this small difference, we impose condition (3.3). Due to the highly nonlinear structure of the underlying SDDE, in order for the SDDE (3.6) to be able to tolerate this small difference without loss of its stability (namely, the perturbed SDDE (3.10) remains stable), we strengthen condition (3.9) to condition (3.8). The following theorem can be proved in the same way as [2, Theorem 3.4] was proved.

Theorem 3.4 *Let Assumptions 3.1, 3.2 and 3.3 hold. Assume also that*

$$\tau \le \frac{\sqrt{2\beta_1\beta_2}}{\beta_4} \wedge \frac{2\beta_1\beta_3}{\beta_4^2}. \tag{3.11}$$

Then for any given initial data (2.2), the solution of the SDDE (2.1) has the properties that

$$\int_0^\infty \mathbb{E}U_1(x(t),t)dt < \infty \tag{3.12}$$

and

$$\sup_{0\le t<\infty} \mathbb{E}U(x(t),r(t),t) < \infty. \tag{3.13}$$

As demonstrated in [2], we can obtain stability results from Theorem 3.4 if we know a bit more on function U_1 or U. For example, if there exists a pair of positive constants c and p such that

$$c|x|^p \le U_1(x,t), \quad \forall (x,t) \in R^n \times R_+, \tag{3.14}$$

then (3.12) implies

$$\int_0^\infty \mathbb{E}|x(t)|^p dt < \infty, \tag{3.15}$$

namely, the SDDE (2.1) is H_∞-stable in L^p. If, furthermore,

$$p \ge 2 \quad \text{and} \quad (p+q_1-1) \vee (p+2q_2-2) \le q, \tag{3.16}$$

(please recall that q_1, q_2 and q were specified in Assumption 3.1 and the standing hypothesis (2.3), respectively), then we can show in the same way as in [2] that

$$\lim_{t\to\infty} \mathbb{E}|x(t)|^p = 0, \tag{3.17}$$

namely, the SDDE (2.1) is asymptotically stable in L^p.

Let us now begin to establish the main result of this paper which shows the almost sure asymptotic stability of the SDDE (2.1) in the sense that $\lim_{t\to\infty} x(t) = 0$ a.s. It should be pointed out that there is no such a result in [2].

Theorem 3.5 *Let the conditions of Theorem 3.4 hold. Assume that there is a non-decreasing function $\mu : R_+ \to R_+$ such that $\lim_{u\to\infty} \mu(u) = \infty$ and*

$$\mu(|x|) \le U(x,i,t), \quad \forall (x,i,t) \in R^n \times S \times R_+. \tag{3.18}$$

Assume moreover that there exists a continuous function $W : R^n \to R_+$ such that

$$W(x) = 0 \textit{ if and only if } x = 0 \tag{3.19}$$

and

$$W(x) \le U_1(x,t), \quad \forall (x,t) \in R^n \times R_+. \tag{3.20}$$

Then for any given initial data (2.2), the solution of the SDDE (2.1) satisfies

$$\lim_{t\to\infty} x(t) = 0 \quad a.s. \tag{3.21}$$

Proof. Fix the initial data (2.2) arbitrarily. By (3.12) and the Fubini theorem, we have

$$E\int_0^\infty U_1(x(t),t)dt < \infty$$

whence

$$\int_0^\infty U_1(x(t),t)dt < \infty \quad a.s.$$

This, along with (3.20), gives

$$\int_0^\infty W(x(t))dt < \infty \quad a.s.$$

This implies

$$\liminf_{t\to\infty} W(x(t)) = 0 \quad a.s. \tag{3.22}$$

But our aim here is to show

$$\lim_{t\to\infty} W(x(t)) = 0 \quad a.s. \tag{3.23}$$

from which can we further obtain our assertion (3.21). As the remaining proof is very technical, we divide it into three steps.

Step 1. Let $k_0 > 0$ be a sufficiently large integer such that $\|\xi\| < k_0$. For each integer $k \geq k_0$, define the stopping time

$$\sigma_k = \inf\{t \geq 0 : |x(t)| \geq k\},$$

where throughout this paper we set $\inf \emptyset = \infty$ (as usual $\emptyset$ denotes the empty set). It is easy to see that σ_k is increasing as $k \to \infty$ and, by our standing hypothesis, $\lim_{k\to\infty} \sigma_k = \infty$ a.s. By the generalised Itô formula (see, e.g., [21, Lemma 1.9 on page 49]), we obtain from (2.5) that

$$\mathbb{E}V(\hat{x}_{t\wedge\sigma_k}, \hat{r}_{t\wedge\sigma_k}, t\wedge\sigma_k) = V(\hat{x}_0, \hat{r}_0, 0) + \mathbb{E}\int_0^{t\wedge\sigma_k} LV(\hat{x}_s, \hat{r}_s, s)ds \tag{3.24}$$

for any $t \geq 0$ and $k \geq k_0$. We now let $\theta = \beta_4^2/(2\beta_1)$. (Please recall that θ is the free parameter in the definition of the Lyapunov functional.) By condition (3.11), we have

$$\theta\tau^2 \leq \beta_2 \quad \text{and} \quad \theta\tau \leq \beta_3. \tag{3.25}$$

By Assumption 3.2, it is also easy to see that

$$\begin{aligned} &U_x(x(t), r(t), t)[f_1(x(t), x(t-\delta(t)), r(t), t) - f_1(x(t), x(t), r(t), t)] \\ &\leq \beta_1 |U_x(x(t), r(t), t)|^2 + \frac{\beta_4^2}{4\beta_1}|x(t) - x(t-\delta(t))|^2. \end{aligned} \tag{3.26}$$

It then follows from (3.4) along with Assumption 3.3 that

$$\begin{aligned} &LV(\hat{x}_s, \hat{r}_s, s) \\ &\leq -\alpha_1 U_1(x(s), s) + \alpha_2 U_1(x(s-\delta(s)), s-\delta(s)) \\ &- \alpha_3 U_2(x(s), s) + \alpha_4 U_2(x(s-\delta(s)), s-\delta(s)) + \frac{\beta_4^2}{4\beta_1}|x(s) - x(s-\delta(s))|^2 \\ &- \frac{\beta_4^2}{2\beta_1}\int_{s-\tau}^s \Big[\tau|f(x(v), x(v-\delta(v)), r(v), v)|^2 + |g(x(v), x(v-\delta(v)), r(v), v)|^2\Big]dv. \end{aligned}$$

Substituting this into (3.24) we can then show

$$\mathbb{E}V(\hat{x}_{t\wedge\sigma_k}, \hat{r}_{t\wedge\sigma_k}, t\wedge\sigma_k) \leq C_1 + H_1(t,k) - H_2(t,k), \tag{3.27}$$

where C_1 is a positive constant and

$$\begin{aligned} H_1(t,k) &= \frac{\beta_4^2}{4\beta_1}\mathbb{E}\int_0^{t\wedge\sigma_k} |x(s) - x(s-\delta(s))|^2 ds, \\ H_2(t,k) &= \frac{\beta_4^2}{2\beta_1}\mathbb{E}\int_0^{t\wedge\sigma_k} \Big(\int_{s-\tau}^s \Big[\tau|f(x(v), x(v-\delta(v)), r(v), v)|^2 \\ &\qquad + |g(x(v), x(v-\delta(v)), r(v), v)|^2\Big]dv\Big)ds. \end{aligned}$$

This along with condition (3.18) implies that

$$\mathbb{E}\mu(|x(t\wedge\sigma_k)|)\le C_1+H_1(t,k)-H_2(t,k),\quad \forall t\ge 0.$$

But

$$\mathbb{E}\mu(|x(t\wedge\sigma_k)|)\ge \mu(k)\mathbb{P}(\sigma_k\le t).$$

Hence

$$\mu(k)\mathbb{P}(\sigma_k\le t)\le C_1+H_1(t,k)-H_2(t,k)$$

Letting $k\to\infty$, we get

$$\limsup_{k\to\infty}\Big(\mu(k)\mathbb{P}(\sigma_k\le t)\Big)\le C_1+\bar H_1(t)-\bar H_2(t),\quad \forall t\ge 0, \tag{3.28}$$

where

$$\begin{aligned}\bar H_1(t)&=\frac{\beta_4^2}{4\beta_1}\int_0^t \mathbb{E}|x(s)-x(s-\delta(s))|^2ds,\\ \bar H_2(t)&=\frac{\beta_4^2}{2\beta_1}\int_0^t \mathbb{E}\Big(\int_{s-\tau}^s \Big[\tau|f(x(v),x(v-\delta(v)),r(v),v)|^2\\ &\qquad\qquad +|g(x(v),x(v-\delta(v)),r(v),v)|^2\Big]dv\Big)ds.\end{aligned}$$

For $t\in[0,\tau]$, we clearly have

$$\bar H_1(t)\le\frac{\beta_4^2}{2\beta_1}\int_0^\tau(\mathbb{E}|x(s)|^2+\mathbb{E}|x(s-\delta(s))|^2)ds\le\frac{\tau\beta_4^2}{\beta_1}\Big(\sup_{-\tau\le v\le\tau}\mathbb{E}|x(v)|^2\Big)=:C_2,$$

where, as usual, =: means ‘denoted by’. For $t>\tau$, we have

$$\bar H_1(t)\le C_2+\frac{\beta_4^2}{4\beta_1}\int_\tau^t \mathbb{E}|x(s)-x(s-\delta(s))|^2ds.$$

But, it follows from the SDDE (2.1) that, for $s\ge\tau$,

$$\begin{aligned}&\mathbb{E}|x(s)-x(s-\delta(s))|^2\\ =&\mathbb{E}\Big|\int_{s-\delta(s)}^s f(x(v),x(v-\delta(v)),r(v),v)dv+\int_{s-\delta(s)}^s g(x(v),x(v-\delta(v)),r(v),v)dB(v)\Big|^2\\ \le&2\mathbb{E}\int_{s-\tau}^s\Big(\tau|f(x(v),x(v-\delta(v)),r(v),v)|^2+|g(x(v),x(v-\delta(v)),r(v),v)|^2\Big)dv.\end{aligned}$$

Hence

$$\begin{aligned}\bar H_1(t)&\le C_2+\frac{\beta_4^2}{2\beta_1}\int_\tau^t \mathbb{E}\Big(\int_{s-\tau}^s \Big[\tau|f(x(v),x(v-\delta(v)),r(v),v)|^2\\ &\qquad\qquad +|g(x(v),x(v-\delta(v)),r(v),v)|^2\Big]dv\Big)ds\\ &=C_2+\bar H_2(t).\end{aligned}$$

In other words, we always have

$$\bar H_1(t)\le C_2+\bar H_2(t),\quad \forall t\ge 0. \tag{3.29}$$

Substituting this into (3.28) yields

$$\limsup_{k\to\infty}\Big(\mu(k)\mathbb{P}(\sigma_k\le t)\Big)\le C_1+C_2,\quad \forall t\ge 0, \tag{3.30}$$

As this holds for any $t \geq 0$, we must have

$$\limsup_{k\to\infty} \Big(\mu(k)\mathbb{P}(\sigma_k < \infty)\Big) \leq C_1 + C_2.$$

Consequently, there exists a k_1 such that

$$\mu(k)\mathbb{P}(\sigma_k < \infty) \leq C_1 + C_2 + 1, \quad \forall k \geq k_1. \tag{3.31}$$

Step 2. In this step we will prove (3.23) by contradiction. For this purpose, we assume that (3.23) were not true. We can then find a number $\varepsilon \in (0, 1/4)$ such that

$$\mathbb{P}(\Omega_1) \geq 4\varepsilon, \tag{3.32}$$

where $\Omega_1 := \{\limsup_{t\to\infty} W(x(t)) > 2\varepsilon\}$. By (3.31), we can find an integer $\kappa > k_1$ such that $\mathbb{P}(\sigma_\kappa < \infty) \leq \varepsilon$. This means that

$$P(\Omega_2) \geq 1 - \varepsilon. \tag{3.33}$$

where $\Omega_2 := \{|x(t)| < \kappa \text{ for } \forall t \geq -\sigma\}$. By (3.32) and (3.33)

$$\mathbb{P}(\Omega_1 \cap \Omega_2) \geq \mathbb{P}(\Omega_1) - \mathbb{P}(\Omega_2^c) \geq 3\varepsilon, \tag{3.34}$$

where Ω_2^c is the complement of Ω_2.

Let us now define the stopped process $z(t) = x(t \wedge \sigma_\kappa)$ for $t \geq -\tau$. Clearly, $z(t)$ is a bounded Itô process with its differential

$$dz(t) = \phi(t)dt + \psi(t)dB(t), \tag{3.35}$$

where

$$\phi(t) = f(x(t), x(t - \delta(t)), t, r(t))I_{[0,\sigma_\kappa)}(t),$$
$$\psi(t) = g(x(t), x(t - \delta(t)), t, r(t))I_{[0,\sigma_\kappa)}(t).$$

By the polynomial growth condition (3.1), we see that $\phi(t)$ and $\psi(t)$ are bounded processes, say

$$|\phi(t)| \vee |\psi(t)| \leq C_3 \quad a.s. \tag{3.36}$$

for all $t \geq 0$. Moreover, we also observe that $|z(t)| \leq \kappa$ for all $t \geq -\tau$. Define a sequence of stopping times

$$\begin{aligned}
\zeta_1 &= \inf\{t \geq 0 : W(z(t)) \geq 2\varepsilon\}, \\
\zeta_{2j} &= \inf\{t \geq \zeta_{2j-1} : W(z(t)) \leq \varepsilon\}, \quad j = 1, 2, \cdots, \\
\zeta_{2j+1} &= \inf\{t \geq \zeta_{2j} : W(z(t)) \geq 2\varepsilon\}, \quad j = 1, 2, \cdots.
\end{aligned}$$

By (3.22) and the definitions of Ω_1 and Ω_2, we have

$$\Omega_1 \cap \Omega_2 \subset \{\sigma_\kappa = \infty\} \bigcap \Big(\cap_{j=1}^\infty \{\zeta_j < \infty\} \Big). \tag{3.37}$$

We also note that for all $\omega \in \Omega_1 \cap \Omega_2$, and $j \geq 1$,

$$W(z(\zeta_{2j-1})) - W(z(\zeta_{2j})) = \varepsilon \text{ and } W(z(t)) \geq \varepsilon \text{ when } t \in [\zeta_{2j-1}, \zeta_{2j}]. \tag{3.38}$$

As $W(x)$ is uniformly continuous in the closed ball $S_\kappa := \{x \in R^n : |x| \leq \kappa\}$, we can find a positive number ρ sufficiently small for which

$$|W(z_1) - W(z_2)| < \varepsilon \text{ whenever } z_1, z_2 \in S_\kappa \text{ with } |z_1 - z_2| < \rho. \tag{3.39}$$

It is useful to highlight that for $\omega \in \Omega_1 \cap \Omega_2$, if $|z(\zeta_{2j-1}+u) - z(\zeta_{2j-1})| < \rho$ for all $u \in [0, \lambda]$ and some $\lambda > 0$, then $\zeta_{2j} - \zeta_{2j-1} \geq \lambda$.

We now observe from (3.12) and condition (3.20) that

$$C_4 := \mathbb{E} \int_0^\infty W(x(t))dt < \infty. \tag{3.40}$$

Choose a sufficiently small positive number λ and then a sufficiently large positive integer j_0 such that

$$2C_3^2 \lambda(\lambda + 4) \leq \varepsilon \rho^2 \quad \text{and} \quad C_4 < \varepsilon^2 \lambda j_0. \tag{3.41}$$

By (3.34) and (3.37), we can further choose a sufficiently large number T for

$$\mathbb{P}(\zeta_{2j_0} \leq T) \geq 2\varepsilon. \tag{3.42}$$

In particular, if $\zeta_{2j_0} \leq T$, $|z(\zeta_{2j_0})| = \varepsilon$ and hence $\zeta_{2j_0} < \sigma_\kappa$ by the definition of $z(t)$ (otherwise $|z(\zeta_{2j})| = |z(\sigma_\kappa)| = \kappa$, a contradiction). We hence have

$$z(t, \omega) = x(t, \omega) \text{ for all } 0 \leq t \leq \zeta_{2j_0} \text{ and } \omega \in \{\zeta_{2j_0} \leq T\}. \tag{3.43}$$

By the Burkholder-Davis-Gundy inequality (see, e.g., [21, Theorem 2.13 on page 70]), we can then derive from (3.35) that, for $1 \leq j \leq j_0$,

$$\begin{aligned}
&\mathbb{E}\Big(\sup_{0 \leq t \leq \lambda} |z(\zeta_{2j-1} \wedge T + t) - z(\zeta_{2j-1} \wedge T)|^2 \Big) \\
\leq& 2\lambda \mathbb{E} \int_{\zeta_{2j-1} \wedge T}^{\zeta_{2j-1} \wedge T + \lambda} |\phi(s)|^2 ds + 8\mathbb{E} \int_{\zeta_{2j-1} \wedge T}^{\zeta_{2j-1} \wedge T + \lambda} |\psi(s)|^2 ds \Big) \\
\leq& 2C_3^2 \lambda(\lambda + 4).
\end{aligned}$$

This, together with (3.41), implies

$$\mathbb{P}\Big(\sup_{0 \leq t \leq \lambda} |z(\zeta_{2j-1} \wedge T + t) - z(\zeta_{2j-1} \wedge T)| \geq \rho \Big) \leq \varepsilon.$$

Noting that $\zeta_{2j-1} \leq T$ if $\zeta_{2j_0} \leq T$, we can derive from (3.42) and the above inequality that

$$\begin{aligned}
&\mathbb{P}\Big(\{\zeta_{2j_0} \leq T\} \cap \Big\{ \sup_{0 \leq t \leq \lambda} |z(\zeta_{2j-1} + t) - z(\zeta_{2j-1})| < \rho \Big\} \Big) \\
=& \mathbb{P}(\zeta_{2j_0} \leq T) - \mathbb{P}\Big(\{\zeta_{2j_0} \leq T\} \cap \Big\{ \sup_{0 \leq t \leq \lambda} |z(\zeta_{2j-1} + t) - z(\zeta_{2j-1})| \geq \rho \Big\} \Big) \\
\geq& \mathbb{P}(\zeta_{2j_0} \leq T) - \mathbb{P}\Big(\sup_{0 \leq t \leq \lambda} |z(\zeta_{2j-1} + t) - z(\zeta_{2j-1})| \geq \rho \Big) \\
\geq& \varepsilon.
\end{aligned}$$

This, together with (3.39) (or what we highlighted just below it), implies easily that

$$\mathbb{P}\Big(\{\zeta_{2j_0} \leq T\} \cap \{\zeta_{2j} - \zeta_{2j-1} \geq \lambda\} \Big) \geq \varepsilon. \tag{3.44}$$

Finally, by (3.40), (3.43) and (3.44), we derive

$$\begin{aligned}
C_4 &\geq \sum_{j=1}^{j_0} \mathbb{E}\Big(I_{\{\zeta_{2j_0} \leq T\}} \int_{\zeta_{2j-1}}^{\zeta_{2j}} W(z(t))dt \Big) \geq \varepsilon \sum_{j=1}^{j_0} \mathbb{E}\Big(I_{\{\zeta_{2j_0} \leq T\}} (\zeta_{2j} - \zeta_{2j-1}) \Big) \\
&\geq \varepsilon\lambda \sum_{j=1}^{j_0} \mathbb{P}\Big(\{\zeta_{2j_0} \leq T\} \cap \{\zeta_{2j} - \zeta_{2j-1} \geq \lambda\} \Big) \Big) \geq \varepsilon^2 \lambda j_0.
\end{aligned}$$

But this contradicts the second inequality in (3.41). Therefore (3.23) must hold.

Step 3. In this final step, we will show our assertion (3.21) from (3.23). If (3.21) were not true, then

$$\varepsilon_1 := \mathbb{P}(\Omega_3) > 0,$$

where $\Omega_3 = \{\limsup_{t\to\infty} |x(t)| > 0\}$. On the other hand, by (3.31), we can find an integer k_2 large enough for $\mathbb{P}(\sigma_{k_2} < \infty) \le 0.5\varepsilon_1$. Let $\Omega_4 = \{\sigma_{k_2} = \infty\}$. Then

$$\mathbb{P}(\Omega_3 \cap \Omega_4) \ge \mathbb{P}(\Omega_3) - \mathbb{P}(\Omega_4^c) \ge 0.5\varepsilon_1.$$

For any $\omega \in \Omega_3 \cap \Omega_4$, $x(t,\omega)$ is bounded on $t \in R_+$. We can then find a sequence $\{t_j\}_{j\ge1}$ such that $t_j \to \infty$ and $x(t_j,\omega) \to \bar{x}(\omega) \neq 0$ as $j \to \infty$. This, together with the continuity of W, implies

$$\lim_{j\to\infty} W(x(t_j,\omega)) = W(\bar{x}(\omega)) > 0.$$

Consequently,

$$\limsup_{t\to\infty} W(x(t,\omega)) > 0 \quad \text{for all } \omega \in \Omega_3 \cap \Omega_4.$$

But this contradicts (3.23). We therefore must have our assertion (3.21). The proof is complete. $\Box$.

4 An Example

In this section we will discuss an example to illustrate our theory due to the page limit. Although our example is a scalar hybrid SDDE, it will illustrate our theory fully.

Example 4.1 Consider a scalar hybrid SDDE

$$dx(t) = f(x(t), x(t-\delta(t)), r(t), t)dt + g(x(t), x(t-\delta(t)), r(t), t)dB(t), \tag{4.1}$$

where $B(t)$ is a scalar Brownian motion, $r(t)$ is a Markov chain on the state space $S = \{1, 2\}$ with its generator

$$\Gamma = \begin{pmatrix} -1 & 1 \\ 10 & -10 \end{pmatrix}, \tag{4.2}$$

and, moreover, the coefficients f and g are defined by

$$f(x,y,1,t) = -y + y^3 - 5x^3,\ \ f(x,y,2,t) = y - \frac{y^3}{2} - 3x^3,\ \ \ g(x,y,1,t) = y^2,\ \ g(x,y,2,t) = 0.5y^2.$$

We will refer to $r(t)$ as the mode of the system. So the system is operated in two modes, 1 and 2. In model 1, the system is described by the SDDE

$$dx(t) = [-x(t-\delta(t)) + x^3(t-\delta(t)) - 5x^3(t)]dt + x^2(t-\delta(t))dB(t), \tag{4.3}$$

while in mode 2

$$dx(t) = [x(t-\delta(t)) - x^3(t-\delta(t))/2 - 3x^3(t)]dt + 0.5x^2(t-\delta(t))dB(t). \tag{4.4}$$

When the system is being operated, it will switch from one SDDE to the other according to the movement of the Markov chain.

Before applying our new theory, we consider two special cases: $\delta(t) = 0$ and $\delta(t) = 2$ for all $t \ge 0$. In the case of $\delta(t) = 0$, the SDDE (4.1) becomes an SDE. We perform a computer simulation with the initial values $x(0) = 3$ and $r(0) = 2$. The sample paths of the Markov chain and the solution of the SDE are shown in Figure 4.1, which indicates that the SDE is

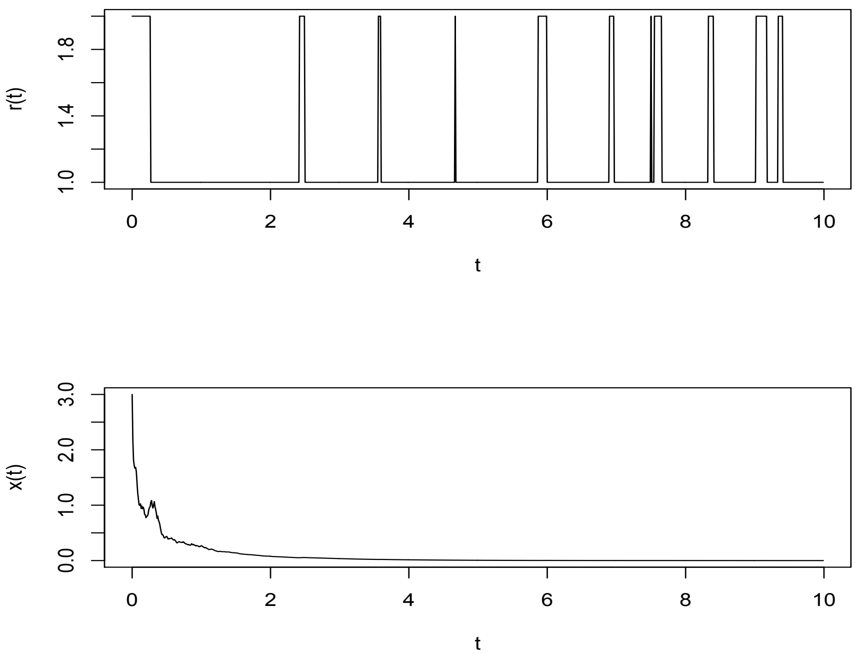

Figure 4.1: The computer simulation of the sample paths of the Markov chain and the SDDE (1.1) with $\delta(t) = 0$ using the Euler–Maruyama method with step size 10^{-4}.

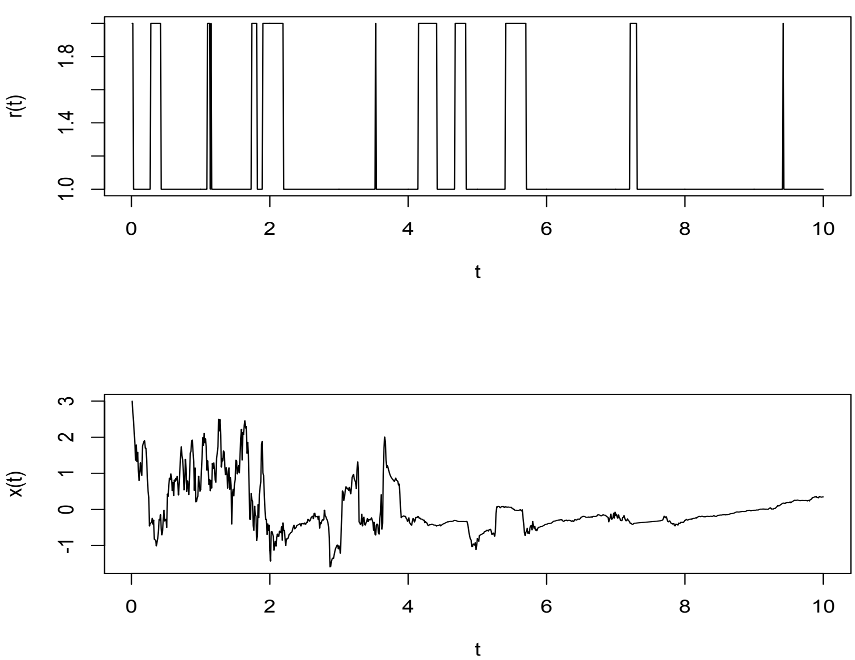

Figure 4.2: The computer simulation of the sample paths of the Markov chain and the SDDE (4.1) with $\delta(t) = 2$ using the Euler–Maruyama method with step size 10^{-4}.

asymptotically stable. In the case of $\delta(t) = 2$, we let the initial data be given by $x(u) = 3+\sin(u)$ for $u \in [-2, 0]$ and $r(0) = 2$ and perform a computer simulation for the sample paths of the Markov chain and the solution of the SDDE. The sample paths are plotted in Figure 4.2, from which we see that the corresponding SDDE is not stable.

These simulations show that when the delay is getting smaller and smaller, the SDDE will become stable. Our theory will be able to show a bound for the delay. For this purpose, let the variable delay $\delta : R_+ \to [0, \tau]$ satisfy the conditions imposed in Section 2 and assume $\bar{\delta} = 0.15$. We first apply [9, Theorem 4.3] to verify our standing hypothesis. Clearly, both coefficients are locally Lipschtiz continuous so Assumption 2.1 in [9] holds. Let $\bar{U}(x) = |x|^6$ for $x \in R$. By the Itô formula,

$$d\bar{U}(x(t)) = \mathbb{L}\bar{U}(x(t), x(t-\delta(t)), r(t), t)dt + 6x^5(t)g(x(t), x(t-\delta(t)), r(t), t)dB(t),$$

where

$$\mathbb{L}\bar{U}(x, y, i, t) = 6x^5 f(x, y, i, t) + 15x^4|g(x, y, i, t)|^2 \tag{4.5}$$

for $(x, y, i, t) \in R \times R \times S \times R_+$. But

$$\mathbb{L}\bar{U}(x, y, 1, t) = 6x^5(-y + y^3 - 5x^3) + 15x^4y^4 \le 5x^6 + y^6 - 18.75x^8 + 9.75y^8 \tag{4.6}$$

and

$$\mathbb{L}\bar{U}(x, y, 2, t) = 6x^5(y - y^3/2 - 3x^3) + (15/4)x^4y^4 \le 5x^6 + y^6 - 14.25x^8 + 3y^8. \tag{4.7}$$

We hence always have

$$\begin{aligned}\mathbb{L}\bar{U}(x, y, i, t) &\le 5x^6 + y^6 - 14.25x^8 + 9.75y^8 \\ &\le 2 + 5x^6 + y^6 - (x^8 + y^8) - 13(1 + x^8) + 11(1 + y^8) \\ &\le c_1 - 13(1 + x^8) + 11(1 + y^8),\end{aligned} \tag{4.8}$$

where

$$c_1 = \sup_{x,y\in R} \big[2 + 5x^6 + y^6 - (x^8 + y^8)\big] < \infty.$$

Therefore, [9, Assumption 4.2] is satisfied. By [9, Theorem 4.3], we can conclude that the SDDE (4.1) with the initial data (2.2) (replace R^n there by R of course) has the unique global solution $x(t)$ on $t \ge -\tau$ and the solution has the property that

$$\sup_{-\tau\le t<\infty} \mathbb{E}|x(t)|^6 < \infty. \tag{4.9}$$

That is, the standing hypothesis (2.3) holds with $q = 6$.

Let us now apply our theorems established in the previous section to discuss the asymptotic stability of the SDDE (4.1). It is easy to see that Assumption 3.1 holds with $q_1 = 3$ and $q_2 = 2$ (and $2(q_1 \vee q_2) = q$). We decompose f as (3.2) with

$$f_1(x, y, 1, t) = -y, \quad f_1(x, y, 2, t) = y, \quad f_2(x, y, 1, t) = y^3 - 5x^3, \quad f_2(x, y, 2, t) = -y^3/2 - 3x^3.$$

Then Assumption 3.2 holds with $\beta_4 = 1$. To verify Assumption 3.3, we define

$$U(x, i, t) = \begin{cases} x^2 + x^4 & \text{if } i = 1, \\ 2x^2 + 3x^4 & \text{if } i = 2 \end{cases} \tag{4.10}$$

for $(x, i, t) \in R \times S \times R_+$. By definition (3.5), we have

$$\mathcal{L}U(x, y, i, t) = \begin{cases} (2x + 4x^3)(-x + y^3 - 5x^3) + (1 + 6x^2)y^4 + x^2 + 2x^4 & \text{if } i = 1, \\ (4x + 12x^3)(x - y^3/2 - 3x^3) + (0.5 + 4.5x^2)y^4 - 10x^2 - 20x^4 & \text{if } i = 2. \end{cases}$$

It is then easy to show

$$\mathcal{L}U(x,y,i,t) \le \begin{cases} -x^2 - 11.5x^4 + 2.5y^4 - 16x^6 + 6y^6 & \text{if } i=1, \\ -6x^2 - 19.5x^4 + 2y^4 - 31.5x^6 + 6y^6 & \text{if } i=2. \end{cases} \tag{4.11}$$

Moreover,

$$|U_x(x,i,t)|^2 = \begin{cases} 4x^2 + 16x^4 + 16x^6 & \text{if } i=1, \\ 16x^2 + 96x^4 + 144x^6 & \text{if } i=2; \end{cases} \tag{4.12}$$

$$|f(x,y,i,t)|^2 = \begin{cases} |-y+y^3-5x^3|^2 \le 3y^2 + 3y^6 + 75x^6 & \text{if } i=1, \\ |y - \frac{1}{2}y^3 - 3x^3|^2 \le 3y^2 + \frac{3}{4}y^6 + 27x^6 & \text{if } i=2; \end{cases} \tag{4.13}$$

$$|g(x,y,i,t)|^2 = \begin{cases} y^4 & \text{if } i=1, \\ 0.25y^4 & \text{if } i=2. \end{cases} \tag{4.14}$$

Setting

$$\beta_1 = 0.1, \quad \beta_2 = 0.05, \quad \beta_3 = 1, \tag{4.15}$$

and using (4.11)-(4.14), we can then show that

$$\begin{aligned} &\mathcal{L}U(x,y,i,t) + \beta_1|U_x(x,i,t)|^2 + \beta_2|f(x,y,i,t)|^2 + \beta_3|g(x,y,i,t)|^2 \\ &\le \begin{cases} -0.6x^2 + 0.15y^2 - 9.9x^4 + 3.5y^4 - 10.65x^6 + 6.15y^6 & \text{if } i=1, \\ -4.4x^2 + 0.15y^2 - 9.9x^4 + 2.25y^4 - 15.75x^6 + 6.1y^6 & \text{if } i=2. \end{cases} \end{aligned} \tag{4.16}$$

This implies

$$\begin{aligned} &\mathcal{L}U(x,y,i,t) + \beta_1|U_x(x,i,t)|^2 + \beta_2|f(x,y,i,t)|^2 + \beta_3|g(x,y,i,t)|^2 \\ &\le -0.6x^2 + 0.15y^2 - 9.9x^4 + 3.5y^4 - 10.65x^6 + 6.15y^6. \\ &\le -0.5(x^2 + 21x^6) + 0.3(y^2 + 21y^6) - 9.9x^4 + 3.5y^4. \end{aligned} \tag{4.17}$$

Letting

$$U_1(x,t) = x^2 + 21x^6,\ U_2(x,t) = x^4,\ \alpha_1 = 0.5,\ \alpha_2 = 0.3,\ \alpha_3 = 9.9,\ \alpha_4 = 3.5, \tag{4.18}$$

we get condition (3.8). Moreover, it is easy to check that condition (3.7) holds as well. In other words, Assumption 3.3 is satisfied. Furthermore, condition (3.11) becomes

$$\tau \le 0.1. \tag{4.19}$$

By Theorem 3.4, we can therefore conclude that the solution of the SDDE (4.1) has the property that

$$\int_0^\infty \mathbb{E}(x^2(t) + x^6(t))dt < \infty. \tag{4.20}$$

It is also easy to see that if we let $\mu(u) = u^2$ and $W(x) = x^2$, then all the conditions of Theorem 3.5 are satisfied too so we also have

$$\lim_{t\to\infty} x(t) = 0 \quad a.s. \tag{4.21}$$

We perform a computer simulation with the time-delay $\delta(t) = 0.1$ for all $t \ge 0$ and the initial data $x(u) = 3 + \sin(u)$ for $u \in [-0.1, 0]$ and $r(0) = 2$. The sample paths of the Markov chain and the solution of the SDDE (4.1) are plotted in Figure 4.3. The simulation supports our theoretical results.

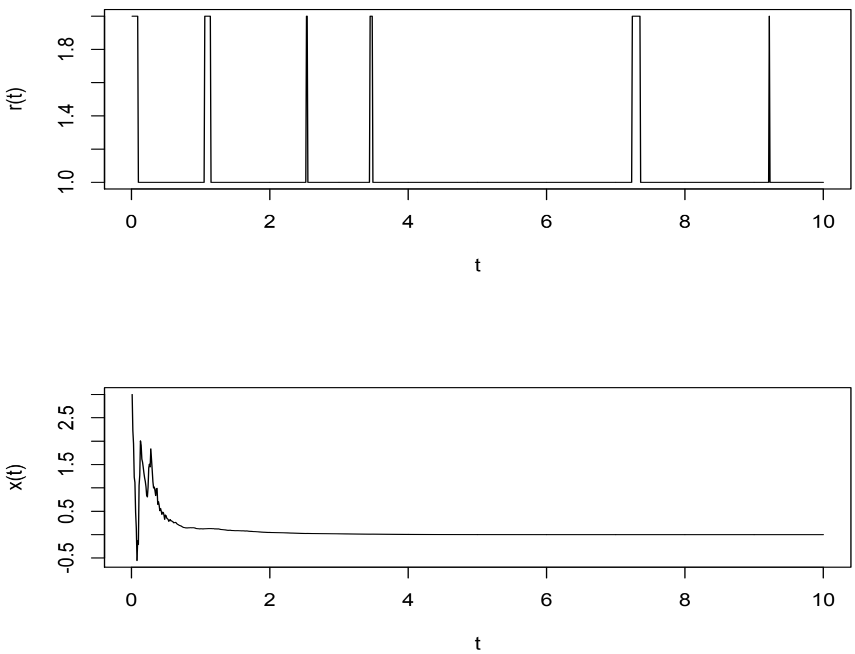

Figure 4.3: The computer simulation of the sample paths of the Markov chain and the SDDE (4.1) with $\delta(t) = 0.1$ using the Euler–Maruyama method with step size 10^{-4}.

5 Conclusion

In this paper we have established the generalised delay-dependent stability criteria for highly nonlinear hybrid SDDEs by removing a restrictive condition imposed in [2]. There are three aspects of the main contributions of the article. First of all, our new criteria do not require the drift coefficient of the underlying SDDE to be globally Lipschitz continuous in the delay component. Thus, the results obtained in this article have much broader applications than those in [2]. Secondly, in addition to the H_∞ stability in L^p and asymptotic stability in L^p, we have discussed the almost sure asymptotic stability which was not studied in [2]. Finally, the methods developed for the proof of the almost sure asymptotic stability are mathematically very technical.

We should also point out that our current work is very much different from that of [10]. Although highly nonlinear hybrid SDDEs are investigated in [10], the stability criteria there are *delay-independent* which are no applicable to *delay-dependent* stable hybrid SDDEs, e.g. the system in Example 4.1. Moreover, Zhu et al. [41] deal with the mean square exponential stability without highly nonlinear coefficients, while Song and Zhu [30] discuss how noise may suppress explosive solutions of differential systems with a new general polynomial growth condition. They are very much different from what we have studied in this paper.

We would also like to mention a number of further research topics. First, the Markovian jump system (MJS) has the limitations in applications, since the jump times of the Markov chain are exponentially distributed with constant transition rates. In comparison, in the semi-Markovian jump system (S-MJS), the jump process is characterised by a fixed matrix of transition probabilities and a matrix of sojourn time probability density functions. Due to the relaxed conditions on the probability distributions, the S-MJS has much broader applications than the MJS (see, e.g., [6, 28]). It is therefore useful to investigate the stability of highly nonlinear

S-MJS. Next, the results in this paper could be generalized to cope with the mode-dependent time-delay systems, in which the derivatives of time delays in some modes could be larger than 1. Moreover, we could replace the Brownian noise by more general noise, e.g., the Lévy noise (see, e.g., [40]) and study the stability of the corresponding highly nonlinear hybrid systems.

Acknowledgements

The authors are very grateful to the referees for their valuable suggestions and comments. The authors would also like to thank the Royal Society (WM160014, Royal Society Wolfson Research Merit Award), the Royal Society and the Newton Fund (NA160317, Royal Society-Newton Advanced Fellowship), the EPSRC (EP/K503174/1), the National Natural Science Foundation of China (11471071, 71571001), the Natural Science Fundation of Shanghai (14ZR1401200), and the Ministry of Education (MOE) of China (MS2014DHDX020) for their financial support.

References

[1] Chen, W., Guan, Z. and Lu, X., Delay-dependent exponential stability of uncertain stochastic systems with multiple delays: an LMI approach, *Systems Control Lett.* 2005; **54**: 547–555.

[2] Fei, W., Hu, L., Mao, X. and Shen, M., Delay dependent stability of highly nonlinear hybrid stochastic systems, *Automatica* 2017; **82**: 165–170.

[3] Fei, W., Hu, L., Mao, X. and Shen, M., Structured robust stability and boundedness of nonlinear hybrid delay systems, *SIAM on Control and Optimization*, in press.

[4] Eguchi, K., Junsing, T., Julsereewong, A., Do, W. and Oota, I., Design of a nesting-type switched-capacitor AC/DC converter using voltage equalizers, *International Journal of Innovative Computing, Information and Control* 2017; **13** (4): 1369–1384.

[5] Ladde, G.S. and Lakshmikantham, V., *Random Differential Inequalities*, Academic Press, 1980.

[6] Li, F., Shi, P., Lim, C. and Wu, L., Fault detection filtering for nonhomogeneous Markovian jump systems via a fuzzy approach, *IEEE Transaction on Fuzzy Systems* 2018; **26** (1): 131–141.

[7] Li, X. and Mao, W., Finite-time H_∞ output tracking control for a class of switched neutral systems with mode-dependent average dwell time method, *International Journal of Innovative Computing, Information and Control* 2017; **13** (3): 767–782.

[8] Hale, J.K. and Lunel, S.M., *Introduction to Functional Differential Equations*, Springer-Verlag, 1993.

[9] Hu, L., Mao, X. and Shen, Y., Stability and boundedness of nonlinear hybrid stochastic differential delay equations, *Systems Control Lett.* 2013; **62**: 178–187.

[10] Hu, L., Mao, X. and Zhang L., Robust stability and boundedness of nonlinear hybrid stochastic differential delay equations, *IEEE Trans. Automat. Control* 2013; **58** (9): 2319—2332.

[11] Ji, Y. and Chizeck, H.J., Controllability, stabilizability and continuous-time Markovian jump linear quadratic control, *IEEE Trans. Automat. Control* 1990; **35**: 777–788.

[12] Kolmanovskii, V.B. and Nosov, V.R., *Stability of Functional Differential Equations*, Academic Press, 1986.

[13] Mao, X., *Stability of Stochastic Differential Equations with Respect to Semimartingales*, Longman Scientific and Technical, 1991.

[14] Mao, X., *Exponential Stability of Stochastic Differential Equations*, Marcel Dekker, 1994.

[15] Mao, X., *Stochastic Differential Equations and Their Applications*, 2nd Edition, Chichester: Horwood Pub., 2007.

[16] Mao, X., Stability of stochastic differential equations with Markovian switching, *Sto. Proc. Appl.* 1999; **79**: 45–67.

[17] Mao, X., Exponential stability of stochastic delay interval systems with Markovian switching, *IEEE Trans. Automat. Control* 2002; **47**(10): 1604–1612.

[18] Mao, X., Stability and stabilization of stochastic differential delay equations, *IET Control Theory & Applications* 2007; **1**(6): 1551–1566.

[19] Mao, X., Lam, J. and Huang, L., Stabilisation of hybrid stochastic differential equations by delay feedback control, *Systems Control Lett.* 2008; **57**: 927–935.

[20] Mao, X., Matasov, A. and Piunovskiy, A.B., Stochastic differential delay equations with Markovian switching, *Bernoulli* 2000; **6**(1):73–90.

[21] Mao, X. and Yuan, C., *Stochastic Differential Equations with Markovian Switching*, Imperial College Press, 2006.

[22] Mariton, M., *Jump Linear Systems in Automatic Control*, Marcel Dekker, 1990.

[23] Mohammed, S.-E.A., *Stochastic Functional Differential Equations*, Longman Scientific and Technical, 1984.

[24] Mohammed, S.-E.A., Stability of linear delay equations under a small noise *Proceedings of the Edinburgh Mathematical Society* 1986; **29**: 233–254.

[25] Niu, Y., Ho, D.W.C. and Lam, J., Robust integral sliding mode control for uncertain stochastic systems with time-varying delay, *Automatica* 2005; **41**: 873–880.

[26] Shaikhet, L., Stability of stochastic hereditary systems with Markov switching, *Theory of Stochastic Processes* 1996; **2**(18): 180–184.

[27] Shen, M., Fei, W., Mao, X. and Liang, Y., Stability of highly nonlinear neutral stochastic differential delay equations, *Systems Control Lett.* 2018 **115**: 1–8.

[28] Shi, P., Li, F., Wu, L. and Lim, C., Neural network-based passive filtering for delayed neutral-type semi-Markovian jump systems, *IEEE Transaction on Neutral Network and Leaning Systems* 2017; **28** (9): 2101–2114.

[29] Shi, P., Mahmoud, M.S., Yi, J. and Ismail, A., Worst case control of uncertain jumping systems with multi-state and input delay information, *Information Sciences* 2006; **176**: 186–200.

[30] Song, S. and Zhu, Q., Noise suppresses explosive solutions of differential systems: A new general polynomial growth condition, *J. Math. Anal. Appl.* 2015; **431**: 648–661.

[31] Stojanovic, S.B. and Debeljkovic, D.Lj., Delay dependent stability of linear time-delay systems, *Theoret. Appl. Mech. TEOPM7* 2013; **40**(2): 223–245.

[32] Su, X., Liu, X., Shi, P. and Yang R., Sliding mode control of discrete-time switched systems with repeated scalar nonlinearities, *IEEE Trans. Automat. Control* 2017; **62** (9): 4604–4610.

[33] Su, X., Liu, X. and Song, Y., Event-triggered sliding-mode control for multi-area power systems, *IEEE Transactions of Industrial Electronics* 2017; **64** (8): 6732–6741.

[34] Su, X., Shi, P., Wu, L. and Song, Y., Fault detection filtering for nonlinear switched stochastic systems, *IEEE Trans. Automat. Control* 2016; **61** (5): 1310–1315.

[35] Wang, B. and Zhu, Q., Stability analysis of Markov switched stochastic differential equations with both stable and unstable subsystems, *Systems Control Lett.* 2017; **105**: 55–61.

[36] Wu, L., Su, X. and Shi, P., Sliding mode control with bounded L_2 gain performance of Markovian jump singular time-delay systems, *Automatica* 2012; **48**(8): 1929–1933.

[37] Xu, S., Lam, J. and Mao, X., Delay-dependent H-infinity control and filtering for uncertain Markovian jump systems with time-varying delays, *IEEE Transactions on Circuits and Systems I* 2007; **54**(9): 2070–2077.

[38] Xu, S., Lam, J., Mao, X. and Y. Zou, A new LMI condition for delay-dependent robust stability of stochastic time-delay systems, *Asian Jornal of Control* 2005; **7**(4): 419–423.

[39] Yue, D. and Han, Q., Delay-dependent exponential stability of stochastic systems with time-varying delay, nonlinearity, and Markovian switching, *IEEE Trans. Automat. Control* 2005; **50**: 217–222.

[40] Zhu, Q., Razumikhin-type theorem for stochastic functional differential equations with Lévy noise and Markov switching, *International Journal of Control* 2017; **90** (8): 1703–1712.

[41] Zhu, Q., Song, S. and Tang, T., Mean square exponential stability of stochastic nonlinear delay systems, *International Journal of Control* 2017; **90** (11): 2384–2393.

[42] Zhu, Q. and Zhang, Q., pth moment exponential stabilisation of hybrid stochastic differential equations by feedback controls based on discrete-time state observations with a time delay, *IET Control Theory Appl.* 2017; **11** (12): 1992–2003.